

# SCHREIER SPECTRUM OF THE HANOI TOWERS GROUP ON THREE PEGS

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**ABSTRACT.** Finite dimensional representations of the Hanoi Towers group are used to calculate the spectra of the finite graphs associated to the Hanoi Towers Game on three pegs (the group serves as a renorm group for the game). These graphs are Schreier graphs of the action of the Hanoi Towers group on the levels of the rooted ternary tree. The spectrum of the limiting graph (Schreier graph of the action on the boundary of the tree) is also provided.

## 1. INTRODUCTION

The spectral theory of graphs and groups has links to many areas of mathematics. It allows for a combinatorial approach to the spectral theory of Laplace-Beltrami operators on manifolds, but at the same time it often gives completely new approach to some problems in algebra, operator algebras, random walks, and combinatorics. It is also closely related to many topics in fractal geometry.

The goal of this note is to show how algebra (more precisely, a group structure) may be used to solve the spectral problem for the sequence of graphs that are modifications of Sierpiński graphs and which naturally arise in the study of the popular combinatorial problem often called Hanoi Towers Game or Hanoi Towers Problem (for a survey of topics and results related to Hanoi Towers Game see [Hin89]).

It is indicated in [GŠ06] and [GŠ07] that the Hanoi Towers Game on  $k$  pegs,  $k \geq 3$ , can be modeled by a self-similar group, denoted  $H^{(k)}$ , generated by a finite automaton on  $k(k-1)/2 + 1$  states over the alphabet  $X = \{0, 1, \dots, k-1\}$  of cardinality  $k$ . The group  $H^{(k)}$  acts by automorphisms on the rooted  $k$ -regular tree and one may consider the sequence of (finite) Schreier graphs  $\{\Gamma_n^{(k)}\}$  (graphs of the action, or orbit graphs) of the action of  $H^{(k)}$  on level  $n$  of the  $k$ -ary tree as well as the (infinite) graph  $\Gamma^{(k)}$  corresponding to the action of  $H^{(k)}$  on the orbit of a point on boundary of the tree (the choice of the point does not play a role).

The combinatorial problem known as Hanoi Towers Game (on  $k$  pegs,  $k \geq 3$ , and  $n$  disks,  $n \geq 0$ ) can be reformulated as the problem of finding the distance (or at least a good asymptotic estimate and/or an algorithm for finding the shortest path) between particular vertices in the graph  $\Gamma_n^{(k)}$  (namely between the vertices  $0^n$  and  $1^n$  in the natural encoding of the vertices at level  $n$  in the  $k$ -ary tree by words of length  $n$  in the alphabet  $X = \{0, 1, \dots, k-1\}$ ). This problem is closely

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related (although not equivalent) to the problem of finding the diameters of the graphs  $\Gamma_n^{(k)}$ ,  $n \geq 0$ .

Questions on distances and diameters are related to the spectral analysis of the involved graphs. It is known that the spectrum of a graph does not determine the graph completely (the famous question of Mark Kac “Can one hear ...” has a negative answer in the context of graphs as well). Nevertheless, the spectrum gives a lot of information about the graph. An upper bound on the diameter can be obtained in terms of the second largest eigenvalue (see [Chu97] for results and references in this direction).

Therefore, one approach to Hanoi Towers Problem is to first try, for fixed  $k$ , to solve the spectral problem for the sequence of graphs  $\{\Gamma_n^{(k)}\}$  and then use the obtained information to compute or provide asymptotics for the diameters and distances between  $0^n$  and  $1^n$  (note that Szegedy has already established such asymptotics by using more direct methods [Sze99]). For  $k = 3$ , the Hanoi Towers Problem has been solved in many ways (both recursive and iterative). Namely, the diameter and the distance between  $0^n$  and  $1^n$  in  $\Gamma_n^{(3)}$  coincide and are equal to  $2^n - 1$ . On the other hand, the problem is open for  $k \geq 4$ . It has long been conjectured (and “proven” many times) that Frame-Stewart algorithm [Fra41, Ste41] does provide an optimal solution. This algorithm takes roughly  $2^{n^{\frac{1}{k-2}}}$  steps and all that can be safely concluded after the work of Szegedy [Sze99] is that this algorithm is asymptotically optimal for  $k \geq 4$ .

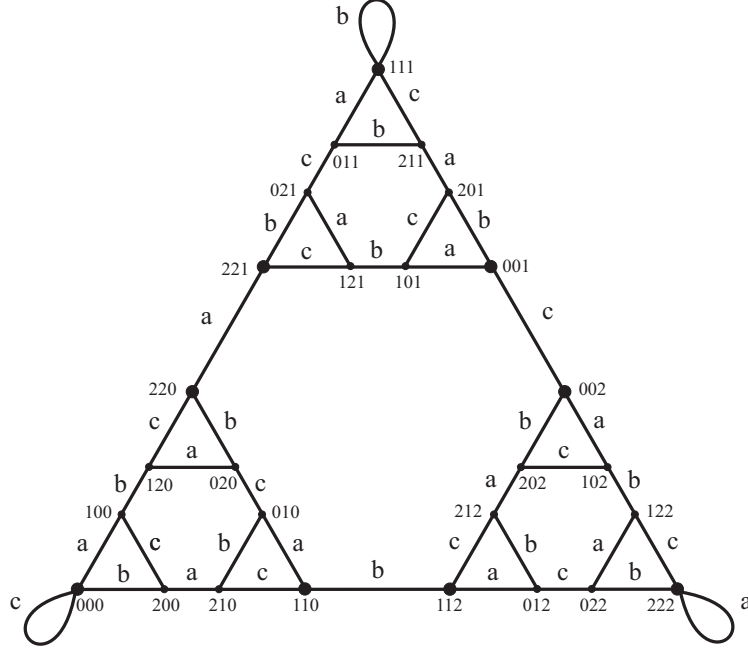
True to form, we only have a full solution to the spectral problem in case  $k = 3$  (announced in [GŠ06]). Some hints on the method used to calculate the spectrum are given in [GŠ07] as well as in [GN07]. Here we give the complete proof and use the opportunity to point out some intricacies of the approach.

In a sense, our approach is algebraic and is based on the fact that Hanoi Towers group  $H^{(k)}$  serves as the renorm group for the model. We would like to stress that  $H^{(k)}$  is not the group of symmetries of the obtained Schreier graph(s), but rather the renorm group of the model (the renorm aspect of self-similar groups is emphasized in the review [Gri07]).

Indeed many regular graphs can be realized as Schreier graphs of some group. The symmetries that are described by the group are not the symmetries of the system but nevertheless may be used to solve the spectral problem. This viewpoint was initiated in [BG00] and further developed in [GŽ01] and [KSS06]. An interesting phenomenon is the appearance of the Schur complement transformation [GŠ07, GN07]. One of the main features of the method is the use of operator recursions coming from the self-similarity structure, followed by algebraic manipulations (sometimes employing  $C^*$ -algebra techniques).

We now describe our main result.

Let  $\Gamma_n$  be the Schreier graph of the Hanoi Towers group  $H^{(3)}$  on three pegs ( $\Gamma_3$  is depicted in Figure 1; see the next section for a precise definition). Further, Let  $\Gamma$  be the orbital Schreier graph of  $H^{(3)}$  corresponding to the orbit of the infinite word  $000\dots$  (this graph is a limit of the sequence of graphs  $\{\Gamma_n\}$ ; see the next section for a precise definition).

FIGURE 1.  $\Gamma_3$ , the Schreier graph of  $H^{(3)}$  at level 3

**Theorem 1.1.** *For  $n \geq 1$ , the spectrum of the graph  $\Gamma_n$ , as a set, has  $3 \cdot 2^{n-1} - 1$  elements and is equal to*

$$\{3\} \cup \bigcup_{i=0}^{n-1} f^{-i}(0) \cup \bigcup_{j=0}^{n-2} f^{-j}(-2),$$

where

$$f(x) = x^2 - x - 3.$$

The multiplicity of the  $2^i$  eigenvalues in  $f^{-i}(0)$ ,  $i = 0, \dots, n-1$  is  $a_{n-i}$  and the multiplicity of the  $2^j$  eigenvalues in  $f^{-j}(-2)$ ,  $j = 0, \dots, n-2$  is  $b_{n-j}$ , where, for  $m \geq 1$ ,

$$a_m = \frac{3^{m-1} + 3}{2}, \quad b_m = \frac{3^{m-1} - 1}{2}.$$

The spectrum of  $\Gamma$  (the Schreier spectrum of  $H^{(3)}$ ), as a set, is equal to

$$\overline{\bigcup_{i=0}^{\infty} f^{-i}(0)}.$$

It consists of a set of isolated points  $I = \bigcup_{i=0}^{\infty} f^{-i}(0)$  and its set of accumulation points  $J$ . The set  $J$  is a Cantor set and is the Julia set of the polynomial  $f$ .

The KNS spectral measure is discrete and concentrated on the set

$$\bigcup_{i=0}^{\infty} f^{-i}\{0, -2\}.$$

The KNS measure of each eigenvalue in  $f^{-i}\{0, -2\}$  is  $\frac{1}{6 \cdot 3^i}$ ,  $i = 0, 1, \dots$ .

Note that here and thereafter, for a map  $\varphi$  and positive integer  $i$ , we denote by  $\varphi^i$  the  $i$ -th iterate  $\varphi \circ \dots \circ \varphi$  of  $\varphi$  under composition. For a set  $A$ ,  $\varphi^{-i}(A)$  denotes the set of inverse images of  $A$  under  $\varphi^i$ .

Before we move into more details, let us mention that, given the relative simplicity of the graphs  $\Gamma_n$ , one may likely obtain the same result by using some other techniques (in particular those developed by Teplyaev and Malozemov [Tep98, MT03]), but the use of self-similar groups brings new ideas to spectral theory that, we believe, could be successfully utilized in many other situations.

As an added benefit, the Hanoi Tower groups  $H^{(k)}$ ,  $k \geq 3$ , are extremely interesting algebraic objects which have some unusual properties (from the point of view of classical group theory). For instance,  $H^{(3)}$  is a finitely generated branch group [GŠ07] that has the infinite dihedral group as a quotient (it is the first known example of a branch group generated by a finite automaton that has this property) and is therefore not a just infinite group (an example of a branch group generated by a finite automaton that has the infinite cyclic group as a quotient was given in [DG07]). Further,  $H^{(3)}$  is amenable but not elementary, and not even subexponentially, amenable group (no familiarity with the amenability property is needed to follow the content of this manuscript; interested reader is referred to [Gre69] and [CSGdlH99] for more background on amenability). The closure of  $H^{(3)}$  in the group of ternary tree automorphisms  $\text{Aut}(X^*)$  is a finitely constrained group (a group of tree automorphisms defined by a finite set of forbidden patterns). Finally, we mention that  $H^{(3)}$  is (isomorphic to) the iterated monodromy group of the post-critically finite rational map  $z \mapsto z^2 - \frac{16}{27z}$  (this map has already been studied from the point of view of complex dynamics [DL05]). Some of the results listed above were announced in [GNŠ06b] and [GNŠ06a] and, along with other new results, will be the subject of a more thorough treatment in a subsequent paper [GNŠ], written jointly with Nekrashevych.

## 2. GROUP THEORETIC MODEL OF HANOI TOWER GAME AND ASSOCIATED SCHREIER GRAPHS

We first describe briefly the Hanoi Towers Game (see [Hin89] for more thorough description and historical details) and then provide a model employing a group action on a rooted  $k$ -ary tree.

Fix an integer  $k$ ,  $k \geq 3$ . The Hanoi Towers Game is played on  $k$  pegs, labeled by  $0, 1, \dots, k-1$ , with  $n$  disks, labeled by  $1, 2, \dots, n$ . All  $n$  disks have different size and the disk labels reflect the relative size of each disk (the disk labeled by 1 is the smallest disk, the disk labeled by 2 is the next smallest, etc). A configuration is (by definition) any placement of the  $n$  disks on the  $k$  pegs in such a way that no disk is below a larger disk (i.e., the size of the disks placed on any single peg is decreasing from the bottom to the top of the peg). In a single step one may move the top disk from one peg to another peg as long as the newly obtained placement of disks is a configuration. Note that this implies that, given two pegs  $x$  and  $y$ , there is only one possible move that involves these two pegs (namely the smaller of the two disks on top of the pegs  $x$  and  $y$  may be moved to the other peg). Initially all disks are on peg 0 and the object of the game is to move all of them to peg 1 in the smallest possible number of steps.

Fix a  $k$ -letter alphabet  $X = \{0, 1, \dots, k-1\}$ . The set  $X^*$  of (finite) words over  $X$  has a rooted  $k$ -ary tree structure. The empty word is the root, and the children

of the vertex  $u$  are the  $k$  vertices  $ux$ , for  $x \in X$ . The set  $X^n$  of words of length  $n$  constitutes level  $n$  in the tree  $X^*$ .

The  $k$ -ary rooted tree  $X^*$  has self-similar structure. Namely, for every vertex  $u$ , the tree  $X^*$  is canonically isomorphic to the subtree  $uX^*$  hanging below vertex  $u$  consisting of vertices with prefix  $u$ . The canonical isomorphism  $\phi_u : X^* \rightarrow uX^*$  is given by  $\phi_u(w) = uw$ , for  $w$  in  $X^*$ .

An automorphism  $g$  of the  $k$ -ary tree  $X^*$  induces a permutation  $\pi_g$  of the alphabet  $X$ , by setting  $\pi_g(x) = g(x)$ , for  $x \in X$ . The permutation  $\pi_g \in \text{Sym}(X)$  is called the root permutation of  $g$ . For a letter  $x$  in  $X^*$ , the section of  $g$  at  $x$  is the automorphism  $g_x$  of  $X^*$  defined by

$$g_x(w) = \phi_{g(x)}^{-1} g \phi_x(w),$$

for  $w$  in  $X^*$ . In other words, the section  $g_x$  of  $g$  at  $x$  acts on the word  $w$  exactly as  $g$  acts on the tail (behind  $x$ ) of the word  $xw$  and we have

$$g(xw) = \pi_g(x) g_x(w),$$

for  $w$  in  $X^*$ . Thus every tree automorphism  $g$  of  $X^*$  can be decomposed as

$$g = \pi_g (g_0, g_1, \dots, g_{k-1}),$$

where  $g_x$ ,  $x = 0, \dots, k-1$  are the sections of  $g$  and  $\pi_g$  is the root permutation of  $X$ . Algebraically, this amounts to decomposition of the automorphism group  $\text{Aut}(X^*)$  of the tree  $X^*$  as the semidirect product  $\text{Sym}(X) \ltimes (\text{Aut}(X^*) \times \dots \times \text{Aut}(X^*))$ . The direct product  $(\text{Aut}(X^*) \times \dots \times \text{Aut}(X^*))$  ( $k$  copies) is the (pointwise) stabilizer of the first level of  $X^*$ . Each factor in the first level stabilizer  $\text{Aut}(X^*) \times \dots \times \text{Aut}(X^*)$  acts on the corresponding subtree of  $X^*$  hanging below the root (the factor corresponding to  $x$  acts on  $xX^*$ ) and the symmetric group  $\text{Sym}(X)$  acts by permuting these  $k$  trees.

For any permutation  $\pi$  in  $\text{Sym}(X)$  define a  $k$ -ary tree automorphism  $a = a_\pi$  by

$$a = \pi(a_0, a_1, \dots, a_{k-1}),$$

where  $a_i = a$  if  $i$  is fixed by  $\pi$  and  $a_i = 1$  otherwise.

For instance, for the transposition  $(ij)$ , the action of  $a_{(ij)}$  on  $X^*$  is given recursively by  $a_{ij}(\emptyset) = \emptyset$  and, for  $x \in X$  and  $w \in X^*$ ,

$$a_{ij}(xw) = \begin{cases} jw, & x = i \\ iw, & x = j \\ xa_{ij}(w), & x \neq i, x \neq j. \end{cases}$$

Thus,  $a_{(ij)}$  “looks” for the first (leftmost) occurrence of one of the letters in  $\{i, j\}$  and replaces it with the other letter. If none of the letters  $i$  or  $j$  appears,  $a_{(ij)}$  leaves the word unchanged.

The configurations in Hanoi Towers Game on  $k$  pegs and  $n$  disks are in bijective correspondence with the words of length  $n$  over  $X$  (vertices at level  $n$  in the  $k$ -ary tree). The word  $x_1 x_2 \dots x_n$  represents the unique configuration in which disk  $i$  is on peg  $x_i$  (once the location of each disk is known, there is only one way to order them on their respective pegs).

The automorphism  $a_{(ij)}$  of  $X^*$  acts on the set of configurations by applying a move between peg  $i$  and peg  $j$ . To apply a move between these two pegs one has to move the smallest disk that is on top of either of these two pegs to the other peg. Let  $w = x_1 x_2 \dots x_n$  be a word over  $X$  and let the leftmost occurrence of one

of the letters  $i$  or  $j$  in  $w$  happen at position  $m$ . This means that none of the disks  $1, 2, \dots, m-1$  is on peg  $i$  or peg  $j$  and disk  $m$  is the smallest disk that appears on either peg  $i$  or  $j$ . Further, disk  $m$  appears on peg  $x_m \in \{i, j\}$  and needs to be moved to peg  $\bar{x}_m$ , where  $\bar{x}_m = j$  if  $x_m = i$  and  $\bar{x}_m = i$  if  $x_m = j$ . The word  $a_{(ij)}(w)$  is obtained from  $w$  by changing the leftmost occurrence of one of the letters  $i$  or  $j$  to the other letter, i.e., by changing the letter  $x_m$  to  $\bar{x}_m$ , which exactly corresponds to the movement of disk  $m$  from peg  $x_m$  to peg  $\bar{x}_m$ . Note that if both peg  $i$  and peg  $j$  are empty there is no occurrence of either  $i$  or  $j$  in  $w$  and in such a case  $a_{(ij)}(w) = w$ , i.e., the configuration is not changed after a move between peg  $i$  and peg  $j$  is applied.

Hanoi Towers group on  $k$  pegs is the group

$$H^{(k)} = \langle \{a_{(ij)} \mid 0 \leq i < j \leq k-1\} \rangle$$

of  $k$ -ary tree automorphisms generated by the automorphisms  $a_{(ij)}$ ,  $0 \leq i < j \leq k-1$ , corresponding to the transpositions in  $\text{Sym}(X)$ . We often drop the superscript in  $H^{(k)}$  in case  $k = 3$ , i.e., we set  $H = H^{(3)}$ . Therefore, if we denote  $a_{(01)} = a$ ,  $a_{(02)} = b$ , and  $a_{(12)} = c$ , the Hanoi Towers group on 3 pegs is the group

$$H = \langle a, b, c \rangle$$

generated by the ternary tree automorphisms  $a$ ,  $b$  and  $c$ , defined by

$$a = (01) (1, 1, a)$$

$$b = (02) (1, b, 1)$$

$$c = (12) (c, 1, 1).$$

The Schreier graph of the action of  $H^{(k)}$  on  $X^*$  on level  $n$  is the regular graph  $\Gamma_n^{(k)}$  of degree  $k(k-1)/2$  whose vertices are the words of length  $n$  over  $X$ , and in which every pair of vertices  $u$  and  $v$  for which  $a_{(ij)}(u) = v$  (or, equivalently,  $a_{(ij)}(v) = u$ ) is connected by an edge labeled by  $a_{(ij)}$ . In other words, two configurations in the Hanoi Towers Game on  $k$  pegs and  $n$  disks are connected by an edge labeled by  $a_{(ij)}$  exactly when they can be obtained from each other by applying a move between peg  $i$  and peg  $j$ . We often drop the superscript in  $\Gamma_n^{(k)}$  in case  $k = 3$ , i.e., we set  $\Gamma_n = \Gamma_n^{(3)}$ . For instance, the graph  $\Gamma_3$  is given in Figure 1. Graphs closely related to  $\Gamma_n^{(k)}$ , usually called Hanoi Towers graphs, are standard feature in many works on Hanoi Tower Game, with a small (but ultimately important) difference. Namely, in our setting, if a word  $w$  does not contain any letter  $i$  or letter  $j$ , there is a loop at  $w$  in  $\Gamma_n^{(k)}$  labeled by  $a_{(ij)}$ . The graph corresponding to Hanoi Towers Game on 3 pegs and 3 disks that appears in [Hin89] (and many other references) does not have loops at the three ‘‘corners’’. Note that our loops do not change the distances (or the diameter) in the graphs, but provide additional level of regularity that is essentially used in our considerations.

The boundary  $\partial X^*$  of the tree  $X^*$  is the set of all right infinite words over  $X$ . Since automorphisms of the tree  $X^*$  preserve the prefixes of words (if two words share a common prefix of length  $m$ , so do their images), the action of any group of tree automorphisms can be naturally extended to an action on the boundary of the tree. The orbit of the word  $0^\infty = 000\dots$  in  $\partial X^*$  under the action of  $H^{(k)}$  consists of all right infinite words that end in  $0^\infty$ . The orbital Schreier graph  $\Gamma^{(k)} = \Gamma_{0^\infty}^{(k)}$  is the countable graph whose vertices are the words in the orbit  $H^{(k)}(0^\infty)$ , and in which a pair of infinite words  $u0^\infty$  and  $v0^\infty$ , with  $|u| = |v|$ , is connected by

an edge labeled by  $a_{(ij)}$  exactly when  $a_{(ij)}(u) = v$  (or, equivalently,  $a_{(ij)}(v) = u$ ). Note that, in terms of the Hanoi Towers Game on  $k$  pegs, the word  $0^\infty$  corresponds to the infinite configuration in which countable many disks labeled by  $1, 2, 3, \dots$  are placed on peg 0, the vertices in  $\Gamma^{(k)}$  are the infinite configurations that can be reached from  $0^\infty$  by using finitely many legal moves and two infinite configurations are connected by an edge labeled by  $a_{(ij)}$  if one can be obtained from the other by applying a move between peg  $i$  and peg  $j$ . When  $k = 3$  we often omit the superscript in  $\Gamma^{(3)}$  and set  $\Gamma = \Gamma^{(3)}$ .

### 3. SCHREIER SPECTRUM OF $H$

The action of  $H$  on level  $n$  induces a permutational  $3^n$ -dimensional representation  $\rho_n : H \rightarrow GL(3^n, \mathbb{C})$  of  $H$ . Denote  $\rho_n(a) = a_n$ ,  $\rho_n(b) = b_n$  and  $\rho_n(c) = c_n$ . The representation  $\rho_n$  can be recursively defined by

$$(3.1) \quad a_0 = b_0 = c_0 = [1]$$

$$a_{n+1} = \begin{bmatrix} 0_n & 1_n & 0_n \\ 1_n & 0_n & 0_n \\ 0_n & 0_n & a_n \end{bmatrix} \quad b_{n+1} = \begin{bmatrix} 0_n & 0_n & 1_n \\ 0_n & b_n & 0_n \\ 1_n & 0_n & 0_n \end{bmatrix} \quad c_{n+1} = \begin{bmatrix} c_n & 0_n & 0_n \\ 0_n & 0_n & 1_n \\ 0_n & 1_n & 0_n \end{bmatrix},$$

where  $0_n$  and  $1_n$  are the zero and the identity matrix, respectively, of size  $3^n \times 3^n$ .

The matrix  $\Delta_n = a_n + b_n + c_n$  is the adjacency matrix of the Schreier graph  $\Gamma_n$  and is defined by

$$\Delta_0 = [3], \quad \Delta_{n+1} = \begin{bmatrix} c_n & 1_n & 1_n \\ 1_n & b_n & 1_n \\ 1_n & 1_n & a_n \end{bmatrix}.$$

The spectrum of  $\Gamma_n$  is the set of  $x$  values for which the matrix

$$\Delta_n(x) = a_n + b_n + c_n - x$$

is not invertible. We introduce another real parameter (besides  $x$ ) and additional operator  $d_n$  defined by

$$d_{n+1} = \begin{bmatrix} 0_n & 1_n & 1_n \\ 1_n & 0_n & 1_n \\ 1_n & 1_n & 0_n \end{bmatrix},$$

and, for  $n \geq 1$ , consider the 2-parameter pencil  $\Delta_n(x, y)$  of  $3^n \times 3^n$  matrices given by

$$\Delta_n(x, y) = a_n + b_n + c_n - x + (y - 1)d_n,$$

i.e.,

$$(3.2) \quad \Delta_n(x, y) = \begin{bmatrix} c - x & y & y \\ y & b - x & y \\ y & y & a - x \end{bmatrix}$$

(observe that here and thereafter we drop the index from  $a_n$ ,  $b_n$ , etc., in order to keep the notation less cumbersome). Instead of trying to determine the values of  $x$  for which  $\Delta_n(x) = \Delta_n(x, 1)$  is not invertible we will find all pairs  $(x, y)$  for which  $\Delta_n(x, y)$  is not invertible (call this set of points in the plane the auxiliary spectrum). This seemingly unmotivated excursion to a higher dimension actually comes naturally (see the comments after Proposition 3.1).

For  $n \geq 1$ , let

$$D_n(x, y) = \det(\Delta_n(x, y)).$$

We provide a recursive formula for the determinant  $D_n(x, y)$ .

**Proposition 3.1.** *We have*

$$D_1(x, y) = -(x - 1 - 2y)(x - 1 + y)^2$$

and, for  $n \geq 2$ ,

$$(3.3) \quad D_n(x, y) = (x^2 - (1 + y)^2)^{3^{n-2}} (x^2 - 1 + y - y^2)^{2 \cdot 3^{n-2}} D_{n-1}(F(x, y)),$$

where  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the 2-dimensional rational map given by

$$F(x, y) = (x', y'),$$

and the coordinates  $x'$  and  $y'$  are given by

$$x' = x + \frac{2y^2(-x^2 + x + y^2)}{(x - 1 - y)(x^2 - 1 + y - y^2)}$$

and

$$y' = \frac{y^2(x - 1 + y)}{(x - 1 - y)(x^2 - 1 + y - y^2)}.$$

*Proof.* By expanding the block matrix (3.2) for  $\Delta_n(x, y)$  one more level (using the recursions for the generators  $a$ ,  $b$  and  $c$  provided in (3.1)) we obtain

$$(3.4) \quad \Delta_n(x, y) = \left[ \begin{array}{ccc|ccc|ccc} c-x & 0 & 0 & y & 0 & 0 & y & 0 & 0 \\ 0 & -x & 1 & 0 & y & 0 & 0 & y & 0 \\ 0 & 1 & -x & 0 & 0 & y & 0 & 0 & y \\ \hline y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\ 0 & y & 0 & 0 & b-x & 0 & 0 & y & 0 \\ 0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\ \hline y & 0 & 0 & y & 0 & 0 & -x & 1 & 0 \\ 0 & y & 0 & 0 & y & 0 & 1 & -x & 0 \\ 0 & 0 & y & 0 & 0 & y & 0 & 0 & a-x \end{array} \right].$$

The last matrix is conjugate, by a permutational matrix that places the entries involving  $c$ ,  $b$  and  $a$  in the last three positions on the diagonal, to the matrix

$$(3.5) \quad \overline{\Delta}_n(x, y) = \left[ \begin{array}{cccccc|ccc} -x & 0 & 0 & y & 1 & 0 & y & 0 & 0 \\ 0 & -x & 1 & 0 & y & 0 & 0 & y & 0 \\ 0 & 1 & -x & 0 & 0 & y & 0 & 0 & y \\ y & 0 & 0 & -x & 0 & 1 & y & 0 & 0 \\ 1 & y & 0 & 0 & -x & 0 & 0 & y & 0 \\ 0 & 0 & y & 1 & 0 & -x & 0 & 0 & y \\ \hline y & 0 & 0 & y & 0 & 0 & c-x & 0 & 0 \\ 0 & y & 0 & 0 & y & 0 & 0 & b-x & 0 \\ 0 & 0 & y & 0 & 0 & y & 0 & 0 & a-x \end{array} \right].$$

Thus, we have

$$\overline{\Delta}_n(x, y) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

where the subdivision into blocks is indicated in (3.5). We can easily calculate the determinant of the matrix  $M_{11}$  (since each block of size  $3^{n-1} \times 3^{n-1}$  in  $M_{11}$  is a scalar multiple of the identity matrix)

$$\det(M_{11}) = (x - 1 - y)^{3^{n-2}} (x + 1 + y)^{3^{n-2}} (x^2 - 1 + y - y^2)^{2 \cdot 3^{n-2}}$$



and the Schur complement ([Zha05] provides a historical overview of Schur complement as well as wide ranging applications) of  $M_{11}$  in  $\overline{\Delta}_n(x, y)$

$$M_{22} - M_{21}M_{11}^{-1}M_{12} = \begin{bmatrix} c - x' & y' & y' \\ y' & b - x' & y' \\ y' & y' & a - x' \end{bmatrix}.$$

Therefore

$$\begin{aligned} D_n(x, y) &= \det(\Delta_n(x, y)) = \det(\overline{\Delta}_n(x, y)) = \det(M_{11}) \det(M_{22} - M_{21}M_{11}^{-1}M_{12}) = \\ &= (x^2 - (1 + y)^2)^{3^{n-2}} (x^2 - 1 + y - y^2)^{2 \cdot 3^{n-2}} D_{n-1}(x', y'). \end{aligned} \quad \square$$

We can indicate now the reason behind the introduction of the new parameter  $y$ . If the same calculation performed during the course of the proof of Proposition 3.1 were applied directly to the matrix  $\Delta_n(x)$ , the corresponding Schur complement would have been equal to

$$\begin{bmatrix} c - x'' & y'' & y'' \\ y'' & b - x'' & y'' \\ y'' & y'' & a - x'' \end{bmatrix},$$

where

$$x'' = x + \frac{2(-x^2 + x + 1)}{(x - 2)(x^2 - 1)} \quad \text{and} \quad y'' = \frac{x}{(x - 2)(x^2 - 1)}.$$

In particular, the blocks off the main diagonal are not identity anymore (as they were in  $\Delta_n(x)$ ). The new parameter  $y$  just keeps track of this change.

The recursion for  $D_n(x, y)$  already gives a good way to calculate the characteristic polynomial of  $\Delta_n$  for small values of  $n$  (it is easier to iterate the recursion 9 times than to try to directly calculate this polynomial for a  $3^{10} \times 3^{10}$  matrix).

Further, for small values of  $n$  we can easily plot the curves in  $\mathbb{R}^2$  along which  $D_n(x, y) = 0$  and get an idea on the structure of the auxiliary spectrum. For instance, for  $n = 1, 2, 3, 4$  these curves are given in Figure 2. It is already apparent from these graphs that with each new iteration several hyperbolae are added to the auxiliary spectrum. The spectrum of  $\Gamma_n$  is precisely the intersection of the line  $y = 1$  and the auxiliary spectrum at level  $n$ . Note that other examples in which the auxiliary spectrum is built from hyperbolae already appear in [BG00] (the first group of intermediate growth, Gupta-Sidki group, Fabrykowski-Gupta group), and there are examples where this is not the case (Basilica group [GŻ02], the iterated monodromy group of the map  $z \mapsto z^2 + i$  [GSŠ07]).

The next step is to provide precise description of the phenomenon we just observed (the appearance of hyperbolae indicates that we may be able to decompose  $D_n(x, y)$  into factors of degree at most 2 over  $\mathbb{R}$ ).

Define a transformation  $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  by

$$\Psi(x, y) = \frac{x^2 - 1 - xy - 2y^2}{y}$$

and a transformation  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = x^2 - x - 3.$$

First, we need some basic understanding of the dynamics of the quadratic map  $f(x)$ . The critical point (the point where  $f'(x) = 0$ ) of the map  $f(x)$  is  $1/2$ . Therefore the critical value  $f(1/2) = -13/4$  is the unique value of  $\theta$  for which  $f(x) = \theta$  has a

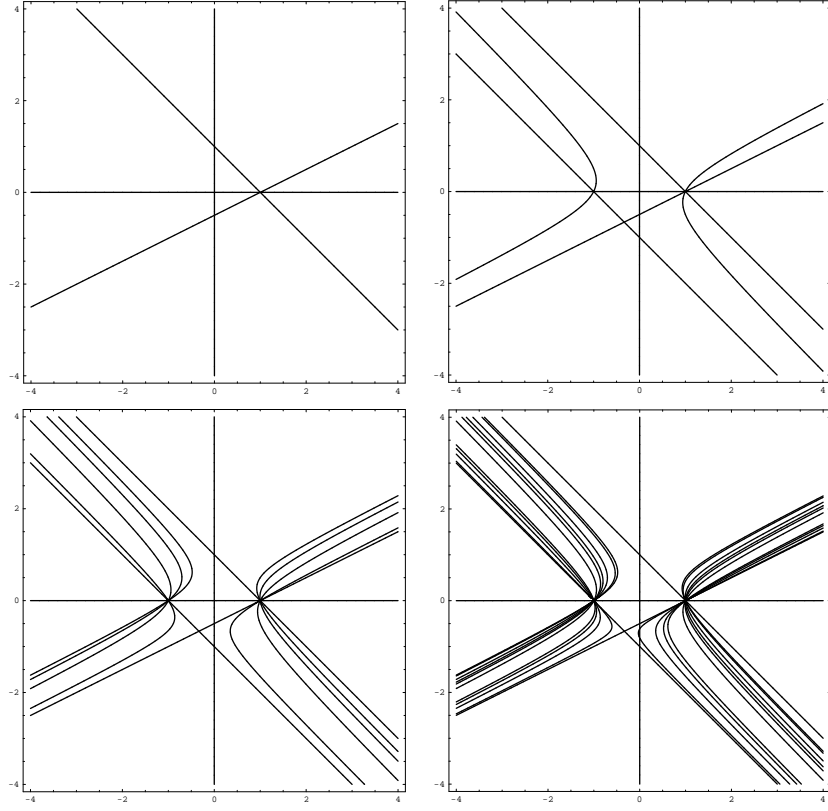


FIGURE 2. Hyperbolae determining the auxiliary spectrum at level 1, 2, 3, and 4

double root. It is easy to check that  $f^{-1}[-2, 3] = [-2, (1 - \sqrt{5})/2] \cup [(1 + \sqrt{5})/2, 3] \subseteq [-2, 3]$ . Since the critical value  $-13/4 \notin f^{-1}[-2, 3] \subseteq [-2, 3]$  it follows that, for any value of  $\theta$  in  $[-2, 3]$ , the entire backward orbit  $f^{-i}(\theta)$  is contained in  $[-2, 3]$  and the sets  $f^{-i}(\theta)$ ,  $i = 0, 1, 2, \dots$ , consist of  $2^i$  real numbers. Further, for such  $\theta$ , the sets  $f^{-i}(\theta)$  are mutually disjoint for  $i = 0, 1, \dots$ , provided  $\theta$  is not a periodic point (a point  $\zeta$  is periodic if  $f^m(\zeta) = \zeta$  for some positive integer  $m$ ).

We come to a simple but crucial observation that lies behind all the calculations that follow. It will eventually allow us to “reduce the dimension” back to 1 and consider iterations of the 1-dimensional polynomial map  $f(x)$  rather than the 2-dimensional rational map  $F(x, y)$ .

**Lemma 3.2.** *The 2-dimensional rational map  $F$  is semi-conjugate to the 1-dimensional polynomial map  $f$  through the map  $\Psi$ , i.e.,*

$$\Psi(F(x, y)) = f(\Psi(x, y))$$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{F} & \mathbb{R}^2 \\ \Psi \downarrow & & \downarrow \Psi \\ \mathbb{R} & \xrightarrow{f} & \mathbb{R} \end{array}$$

*Proof.* The claim can be easily verified directly.  $\square$

Let

$$\begin{aligned}\Psi_\theta(x, y) &= x^2 - 1 - xy - 2y^2 - \theta y = y(\Psi(x, y) - \theta), \\ L(x, y) &= x - 1 - y, \\ K(x, y) &= x^2 - 1 + y - y^2, \\ A_1(x, y) &= x - 1 + y.\end{aligned}$$

In order to simplify notation we sometimes write  $P$  and  $P'$  instead of  $P(x, y)$  and  $P(F(x, y))$ .

**Lemma 3.3.** *Let  $\theta \in [-2, 3]$  and let  $\theta_0$  and  $\theta_1$  be the two distinct real roots of  $f(x) = \theta$ . Then*

$$\frac{A_1}{LK} \Psi_{\theta_0} \Psi_{\theta_1} = \Psi'_\theta.$$

*Proof.* We have

$$\Psi'_\theta = y'(\Psi' - \theta) = \frac{y^2 A_1}{LK} (f(\Psi) - \theta) = \frac{y^2 A_1}{LK} (\Psi - \theta_0)(\Psi - \theta_1) = \frac{A_1}{LK} \Psi_{\theta_0} \Psi_{\theta_1}. \quad \square$$

Define the polynomial

$$D_0(x, y) = -(x - 1 - 2y),$$

two families of polynomials

$$A_n(x, y) = \begin{cases} x - 1 + y, & n = 1 \\ \prod_{\theta \in f^{-(n-2)}(0)} \Psi_\theta, & n \geq 2 \end{cases}$$

and

$$B_n(x, y) = \begin{cases} x + 1 + y, & n = 2 \\ \prod_{\theta \in f^{-(n-3)}(-2)} \Psi_\theta, & n \geq 3 \end{cases},$$

and two integer sequences

$$a_n = \frac{3^{n-1} + 3}{2}, \quad b_n = \frac{3^{n-1} - 1}{2},$$

for  $n \geq 1$ .

Note that each factor  $\Psi_\theta$  that appears above in  $A_n$  and  $B_n$  is quadratic polynomial in  $\mathbb{R}[x, y]$ . It will be shown that the hyperbolae  $\Psi_\theta = 0$  are precisely the hyperbolae in the auxiliary spectrum.

**Lemma 3.4.**

$$\begin{aligned}D'_0 &= \frac{D_0}{L} A_1, \\ A'_1 &= \frac{A_1}{K} A_2, \\ A'_n &= \left( \frac{A_1}{LK} \right)^{2^{n-2}} A_{n+1}, \text{ for } n \geq 2, \\ B'_2 &= \frac{B_2}{K} B_3, \\ B'_n &= \left( \frac{A_1}{LK} \right)^{2^{n-3}} B_{n+1}, \text{ for } n \geq 3.\end{aligned}$$

*Proof.* From Lemma 3.3 we obtain, for  $n \geq 2$ ,

$$A'_n = \prod_{\theta \in f^{-(n-2)}(0)} \Psi'_\theta = \left(\frac{A_1}{LK}\right)^{2^{n-2}} \prod_{\theta \in f^{-(n-1)}(0)} \Psi_\theta = \left(\frac{A_1}{LK}\right)^{2^{n-2}} A_{n+1}.$$

The claim involving  $B'_n$  can be verified in a similar manner. All the other claims do not involve  $n$  and can be easily verified directly.  $\square$

At this moment we can provide a factorization of the determinant  $D_n(x, y)$ .

**Proposition 3.5.**

$$(3.6) \quad \begin{aligned} D_1(x, y) &= D_0 A_1^{a_1}, \\ D_n(x, y) &= D_0 A_1^{a_n} A_2^{a_{n-1}} \dots A_n^{a_1} B_2^{b_n} B_3^{b_{n-1}} \dots B_n^{b_2}, \text{ for } n \geq 2. \end{aligned}$$

*Proof.* The claim is correct for  $n = 1$ . Assume the claim is correct for  $n - 1$  (where  $n \geq 2$ ). Then, by Proposition 3.1,

$$\begin{aligned} D_n(x, y) &= (B_2 L)^{3^{n-2}} K^{2 \cdot 3^{n-2}} D_{n-1}(F(x, y)) = \\ &= (B_2 L)^{3^{n-2}} K^{2 \cdot 3^{n-2}} D'_0 (A'_1)^{a_{n-1}} \dots (A'_{n-1})^{a_1} (B'_2)^{b_{n-1}} \dots (B'_{n-1})^{b_2} = \\ &= (B_2 L)^{3^{n-2}} K^{2 \cdot 3^{n-2}} \frac{D_0 A_1}{L} \left(\frac{A_1}{LK}\right)^{m_n} \frac{A_1^{a_{n-1}}}{K^{a_{n-1}}} A_2^{a_{n-1}} \dots A_n^{a_1} \frac{B_2^{b_{n-1}}}{K^{b_{n-1}}} B_3^{b_{n-1}} \dots B_n^{b_2}, \end{aligned}$$

where  $m_n = (a_{n-2} + 2a_{n-3} + \dots + 2^{n-3}a_1) + (b_{n-2} + 2b_{n-3} + \dots + 2^{n-4}b_2)$ . It is easy to verify that

$$\begin{aligned} 3^{n-2} &= m_n + 1, & a_n &= m_n + a_{n-1} + 1, \\ 2 \cdot 3^{n-2} &= m_n + a_{n-1} + b_{n-1}, & b_n &= b_{n-1} + 3^{n-2}, \end{aligned}$$

and therefore  $D_n(x, y) = D_0 A_1^{a_n} A_2^{a_{n-1}} \dots A_n^{a_1} B_2^{b_n} B_3^{b_{n-1}} \dots B_n^{b_2}$ .  $\square$

*Proof of Theorem 1.1.* A factorization of  $D_n(x, y)$  is already provided in Proposition 3.5. The eigenvalues of  $\Gamma_n$  correspond to the zeros of the polynomial  $D_n(x, 1)$ . The claims in Theorem 1.1 concerning the eigenvalues of  $\Gamma_n$  follow immediately once it is observed that

$$\begin{aligned} \Psi(x, 1) &= f(x), \\ \Psi_\theta(x, 1) &= f(x) - \theta, \\ D_0(x, 1) &= -(x - 3), \\ A_1(x, 1) &= x, \\ A_n(x, 1) &= \prod_{\theta \in f^{-(n-2)}(0)} (f(x) - \theta), \quad n \geq 2 \\ B_2(x, 1) &= x + 2, \\ B_n(x, 1) &= \prod_{\theta \in f^{-(n-3)}(-2)} (f(x) - \theta), \quad n \geq 3. \end{aligned}$$

Further, the forward orbit of 0 under  $f$  escapes to  $\infty$ . Thus 0 is not periodic point and this implies that the sets  $f^{-i}(0)$  are mutually disjoint for  $i = 0, 1, 2, \dots$ . Similarly, since  $f(-2) = 3$  and 3 is fixed point of  $f$ , the point  $-2$  is not periodic and the sets  $f^{-i}(-2)$  are mutually disjoint for  $i = 0, 1, 2, \dots$ . Thus the number of distinct eigenvalues of  $\Gamma_n$ , for  $n \geq 1$ , is  $1 + (2^n - 1) + (2^{n-1} - 1) = 3 \cdot 2^{n-1} - 1$ .

Since  $H$  is amenable (or more obviously, since the graph  $\Gamma$  is amenable), the spectrum of  $\Gamma$  is given by (see [BG00] for details)

$$\overline{\{3\} \cup \bigcup_{i=0}^{\infty} f^{-i}(0) \cup \bigcup_{i=0}^{\infty} f^{-i}(-2)}.$$

Recall that a periodic point  $\zeta$  of  $f$  is repelling if  $|f'(\zeta)| > 1$  (see [Dev89] or [Bea91] for basic notions and results on iteration of rational functions). Since  $f(3) = 3$  and  $f'(3) = 5$ , the point 3 is a repelling fixed point for the polynomial  $f$ . This implies that the backward orbit of 3, which is equal to  $\{3\} \cup \bigcup_{i=0}^{\infty} f^{-i}(-2)$ , is in the Julia set  $J$  of  $f$  (see [Bea91, Theorem 6.4.1]). On the other hand 0 is not in the Julia set (its forward orbit escapes to  $\infty$ ) and therefore the set  $I = \bigcup_{i=0}^{\infty} f^{-i}(0)$  is a countable set of isolated points that accumulates to the Julia set  $J$  of  $f$ . The spectrum of  $\Gamma$ , as a set, is therefore equal to

$$\overline{\bigcup_{i=0}^{\infty} f^{-i}(0)}.$$

The Julia set  $J$  has the structure of a Cantor set since  $f$  is conjugate to the map  $x \mapsto x^2 - 15/4$  and  $-15/4 < -2$  (see [Dev89, Section 3.2]).

Recall that (see [BG00, GZ04]) the KNS spectral measure  $\nu$  of is limit of the counting measures  $\nu_n$  defined for  $\Gamma_n$  ( $\nu_n(B) = m_n(B)/3^n$ , where  $m_n(B)$  counts, including the multiplicities, the eigenvalues of  $\Gamma_n$  in  $B$ ). For the eigenvalues in  $f^{-i}(0)$  we have

$$\lim_{n \rightarrow \infty} \frac{a_{n-i}}{3^n} = \lim_{n \rightarrow \infty} \frac{3^{n-i-1} + 3}{2 \cdot 3^n} = \frac{1}{6 \cdot 3^i}.$$

Since  $b_n = a_n - 2$ , for  $n \geq 1$ , the density of the eigenvalues in  $f^{-i}(-2)$  is also  $1/(6 \cdot 3^i)$ . Since all these densities add up to 1, the KNS spectral measure is discrete and concentrated at these eigenvalues.  $\square$

**Corollary 3.6.** *The characteristic polynomial  $P_n(x)$  of the matrix  $\Delta_n$  decomposes into irreducible factors over  $\mathbb{Q}[x]$  as follows:*

$$\begin{aligned} P_0(x) &= -(x-3) \\ P_1(x) &= -(x-3)x^2 \\ P_2(x) &= -(x-3)x^3 (f(x))^2 & (x+2) \\ P_3(x) &= -(x-3)x^6 (f(x))^3 (f^2(x))^2 & (x+2)^4 g(x+2) \\ P_4(x) &= -(x-3)x^{15} (f(x))^6 (f^2(x))^3 (f^3(x))^2 (x+2)^{13} (g(x+2))^4 g^2(x+2) \\ &\dots \end{aligned}$$

and in general

$$P_n(x) = -(x-3) x^{a_n} (f(x))^{a_{n-1}} \dots (f^{n-1}(x))^{a_1} (x+2)^{b_n} (g(x+2))^{b_{n-1}} \dots g^{n-2}(x+2)^{b_2},$$

where  $g(x) = x^2 - 5x + 5$ .

*Proof.* Observe that  $g(x+2) = f(x) + 2$  (i.e.,  $f$  and  $g$  are conjugate by translation by 2). This implies that, for  $n \geq 1$ ,  $g^n(x+2) = f^n(x) + 2$ . The correctness of the factorization directly follows from Proposition 3.5 (for  $y = 1$ ).

Since  $f(x)$  and  $g(x)$  are irreducible quadratic polynomials and their discriminants are odd, it follows that, for all  $n \geq 1$ ,  $f^n(x)$  and  $g^n(x)$  (and therefore also  $g^n(x+2)$ ) are irreducible (see [AM00, Theorem 2]).  $\square$

Before we conclude the section, it is perhaps necessary to make the following technical remark. During the course of the proof of Proposition 3.1 we used the formula  $D_n = \det(M_{11}) \det(M_{22} - M_{21}M_{11}^{-1}M_{12})$  to establish a recursion for  $D_n$ . However, this formula may be used only when  $M_{11}$  is invertible (thus, we must stay off the lines  $x - 1 - y = 0$ ,  $x + 1 + y = 0$  and the hyperbola  $x^2 - 1 + y - y^2$ ). The problem is compounded by the fact that the formula for  $D_n(x, y)$  needs to be iterated, which then means that it may not be used at points on the pre-images of the above curves under the 2-dimensional map  $F$ . Nevertheless, the factorization formulas (3.6) provided in Proposition 3.5 are correct on some open set in the plane (avoiding these pre-images) and since these formulas represents equalities of polynomials in two variables they are correct on the whole plane.

#### 4. SOME ADDITIONAL COMMENTS

We used the spectra of the level Schreier graphs  $\Gamma_n$  to calculate the spectrum of the boundary Schreier graph  $\Gamma_{0\infty}$  and we called this spectrum the Schreier spectrum of  $H$ . Since the boundary  $\partial X^*$  is uncountable and  $H$  is countable, there are uncountably many orbits of the action of  $H$  on the boundary and there are clearly non-isomorphic ones among them (for instance  $\Gamma_{0\infty}$  has exactly one loop, while  $\Gamma_{(012)\infty}$  has none). However, the spectra of all these boundary Schreier graphs coincide and are equal to the closure of the union of the spectra of  $\Gamma_n$ . This follows from [BG00, Proposition 3.4], since  $H$  acts transitively on each level of the tree  $X^*$  and is amenable (the amenability of groups generated by bounded automata is proven in [BKNV05]).

The boundary  $\partial X^*$  supports a canonical invariant measure  $\mu$  defined as the Bernoulli product measure on  $\partial X^*$  induced by the uniform measure on the finite set  $X$ . Thus we can associate to any group  $G$  of tree automorphisms a unitary representation  $\rho$  of  $G$  on  $L_2(\partial X^*, \mu)$ , defined by  $(\pi(g)(\alpha))(w) = \alpha(g^{-1}(w))$ , for  $g \in G$ ,  $\alpha \in L_2(\partial X^*, \mu)$  and  $w \in \partial X^*$ . If  $S = S^{-1}$  is a finite symmetric set of generators of  $G$ , we associate to the representation  $\pi$  a Hecke type operator defined by  $T_S = \frac{1}{|S|} \sum_{s \in S} \pi(s)$ . It follows from [BG00, Theorem 3.6] and the amenability of  $H$  that, when  $G = H$  and  $S = \{a, b, c\}$ , the spectrum of  $T_S$  is equal to the Schreier spectrum of  $H$  re-scaled by  $1/3$ .

Speaking of amenability, the question of amenability of  $H^{(k)}$  is open for  $k \geq 4$ . The fact that many questions, including diameter, spectra, average distance (see [HS90]) can be answered in case  $k = 3$ , but are still open in case  $k \geq 4$  may very well be related to the possibility that the groups  $H^{(k)}$  are not amenable for  $k \geq 4$ , even though the graphs  $\Gamma^{(k)}$  are amenable (an obvious sequence  $\{F_n\}$  of Følner sets is obtained by declaring  $F_n$  to consist of all right infinite words in which all symbols after position  $n$  are equal to 0).

Thus, both the Hanoi Towers groups  $H^{(k)}$ ,  $k \geq 4$ , and the associated Schreier graphs are very interesting objects and, in our opinion, more work on questions related to algebraic and combinatorial properties of the groups and analysis and random walks on these groups and graphs is certainly needed.

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